The KT Difference Scheme for 1D Lagrangian Gas Dynamics

$$
\begin{cases}
\dfrac{\partial \rho \cdot g}{\partial t} = 0 \\
\dfrac{\partial \rho \cdot g \cdot u}{\partial t} + \dfrac{\partial p}{\partial \xi} = 0 \\
\dfrac{\partial}{\partial}(\rho \cdot g \cdot (\varepsilon + \dfrac{u^2}{2})) + \dfrac{\partial p \cdot u}{\partial \xi} = 0 \\
\dfrac{\partial g}{\partial t} - \dfrac{\partial u}{\partial \xi} = 0 \\
\dot{x} = u
\end{cases}
$$

Difference Scheme for 1D Lagrangian Gas Dynamics I

$$\frac{\partial m_k}{\partial t} = 0$$

$$\frac{\partial I_k}{\partial t} + \frac{1}{h}(F^I_{k+1/2} - F^I_{k-1/2}) = 0$$

$$\frac{\partial E_k}{\partial t} + \frac{1}{h}(F^E_{k+1/2} - F^E_{k-1/2}) = 0$$

$$\frac{\partial g_{k+1/2}}{\partial t} + \frac{1}{h}(F^g_{k+1} - F^g_k) = 0$$

Difference Scheme for 1D Lagrangian Gas Dynamics II

$$F^{I}_{k+1/2} = 0.5\cdot(p^{+}_{k+1/2} + p^{-}_{k+1/2}) - 0.5\cdot c\cdot(I^{+}_{k+1/2} - I^{-}_{k+1/2})$$

$$F^{I}_{k+1/2} = 0.5\cdot((p\cdot u)^{+}_{k+1/2} + (p\cdot u)^{-}_{k+1/2}) - 0.5\cdot c\cdot(E^{+}_{k+1/2} - E^{-}_{k+1/2})$$

$$F^{g}_{k+1} = 0.5\cdot((-u)^{+}_{k+1} + (-u)^{-}_{k+1}) - 0.5\cdot c\cdot(g^{+}_{k+1/2} - g^{-}_{k+1/2})$$

Difference Scheme for 1D Lagrangian Gas Dynamics III

$$I^{+}_{k+1/2} = I_{k+1} - \frac{h}{2} \cdot DI_{k+1}$$

$$I^{-}_{k+1/2} = I_{k} + \frac{h}{2} \cdot DI_{k}$$

$$E^{+}_{k+1/2} = E_{k+1} - \frac{h}{2} \cdot DE_{k+1}$$

$$E^{-}_{k+1/2} = E_{k} + \frac{h}{2} \cdot DE_{k}$$

$$g^{+}_{k} = g_{k+1/2} - \frac{h}{2} \cdot Dg_{k+1/2}$$

$$g^{-}_{k} = g_{k-1/2} + \frac{h}{2} \cdot Dg_{k-1/2}$$

Difference Scheme for 1D Lagrangian Gas Dynamics IV

$$DI_k = \theta \cdot \min\bmod\left(\frac{I_{k+1} - I_k}{h}, \frac{I_k - I_{k-1}}{h}\right)$$

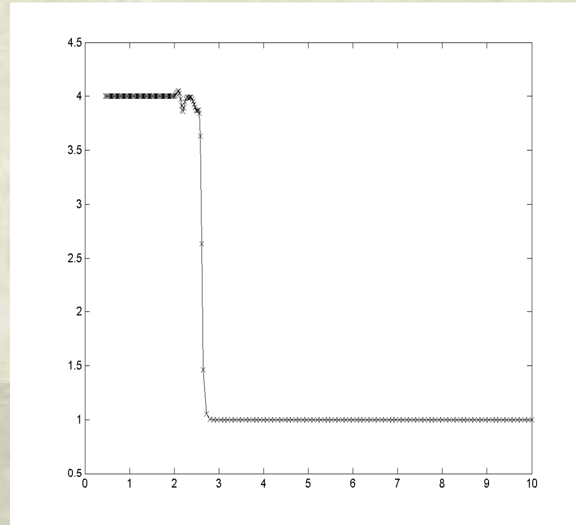
$$DE_k = \theta \cdot \min\bmod\left(\frac{E_{k+1} - E_k}{h}, \frac{E_k - E_{k-1}}{h}\right)$$

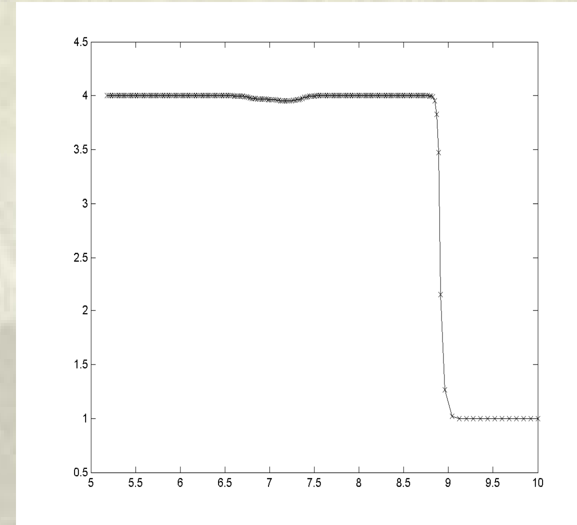
$$Dg_{k+1/2} = \theta \cdot \min\bmod\left(\frac{g_{k+3/2} - g_{k+1/2}}{h}, \frac{g_{k+1/2} - g_{k-1/2}}{h}\right)$$

A test problem for the lagrangian difference scheme

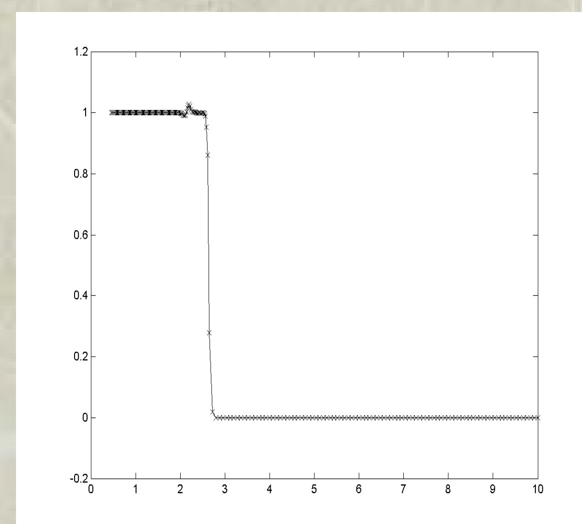
- In the next five slides the results of a test problem numerical modeling are presented
- The initial conditions are: the first region – density 4, specific internal energy – 0.5, velocity – 1; the second region – density – 1, specific internal energy – 1e-6, velocity – 0.
- Boundary conditions: the left boundary – piston with velocity 1, the right one – rigid wall.
- The plots present density and velocity for 10 successive times.

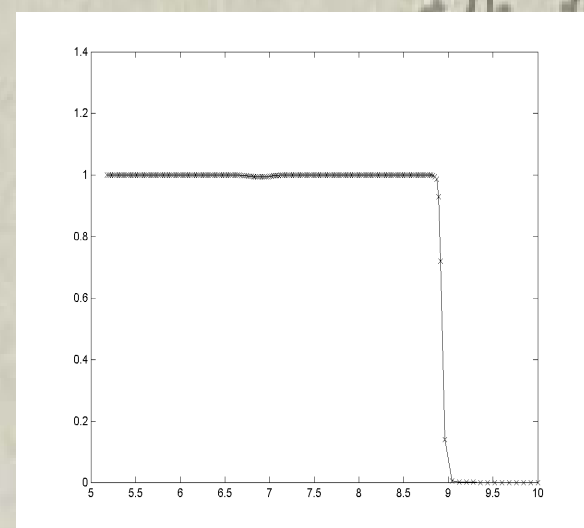
density



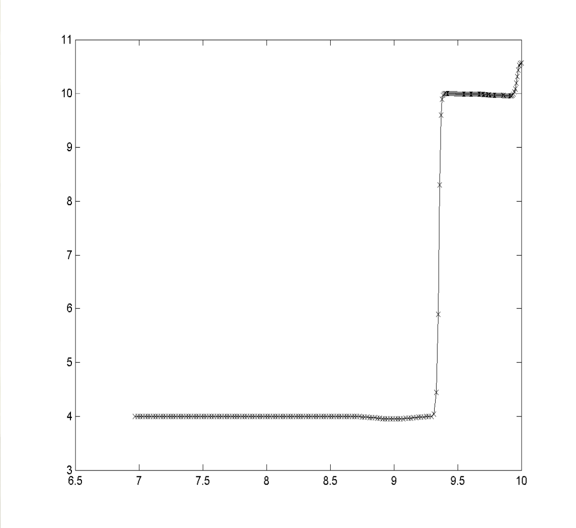


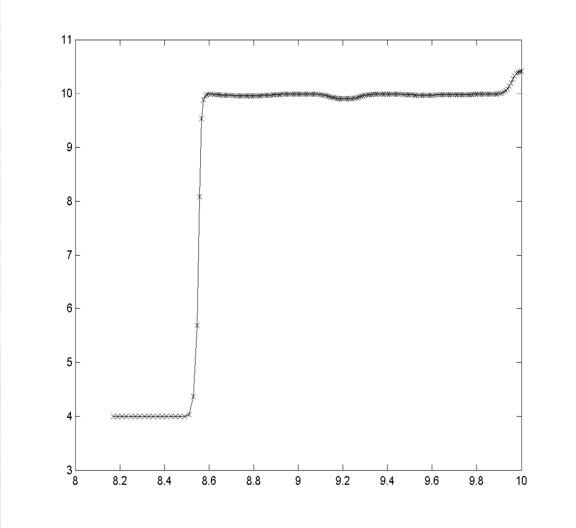
velocity



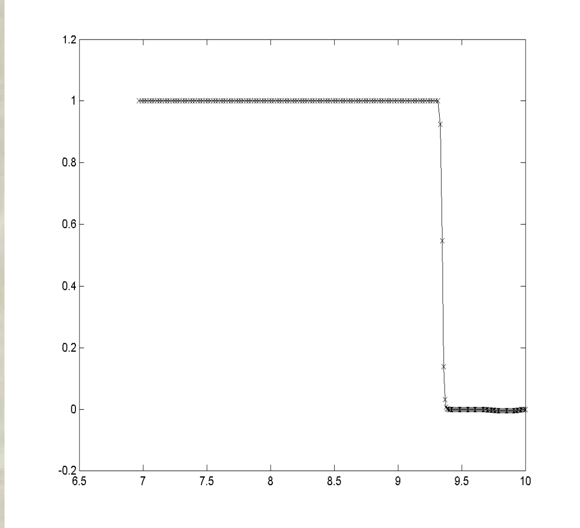


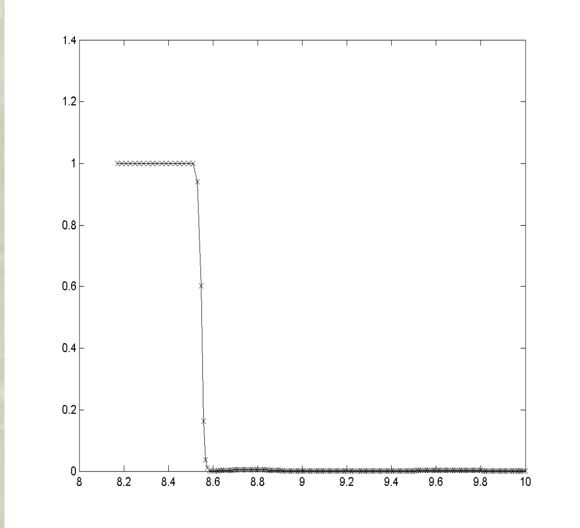
density



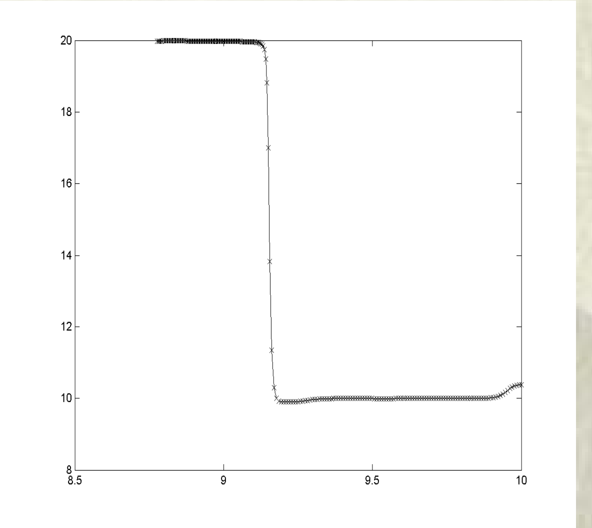


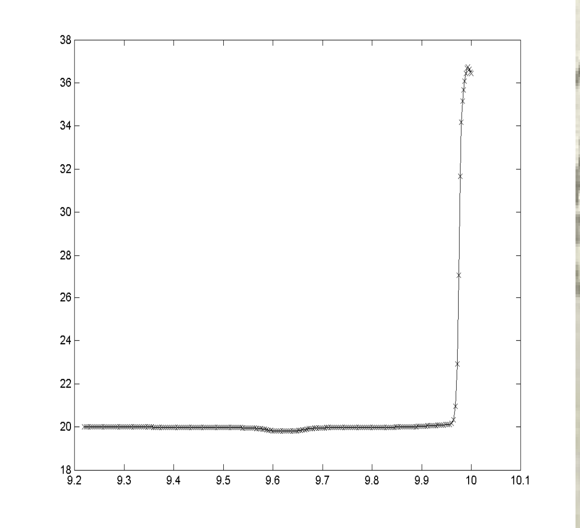
velocity



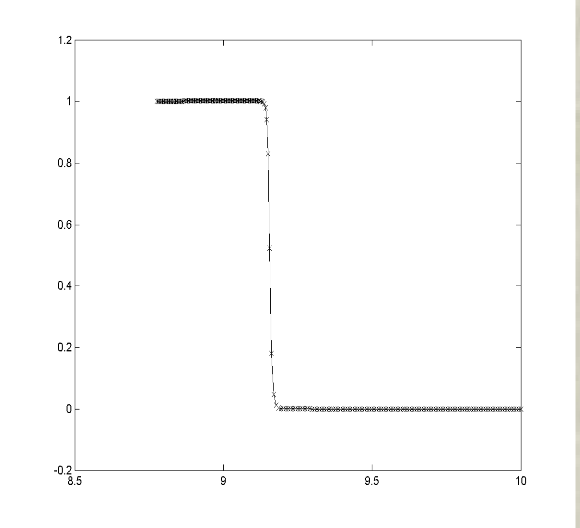


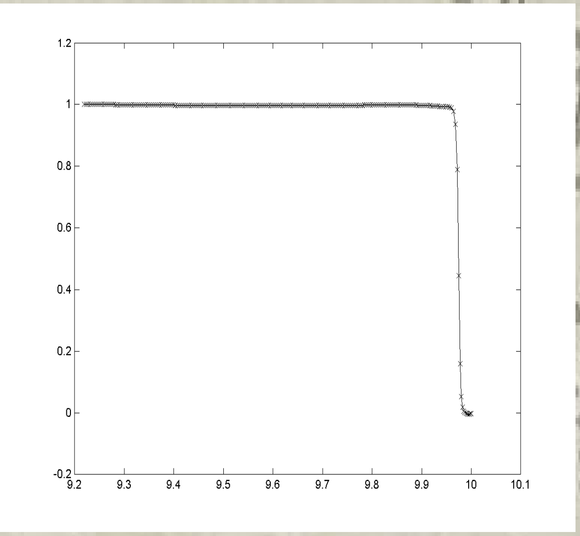
density



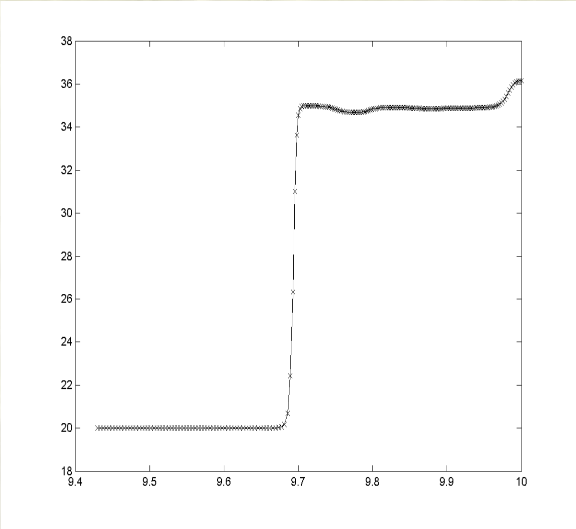


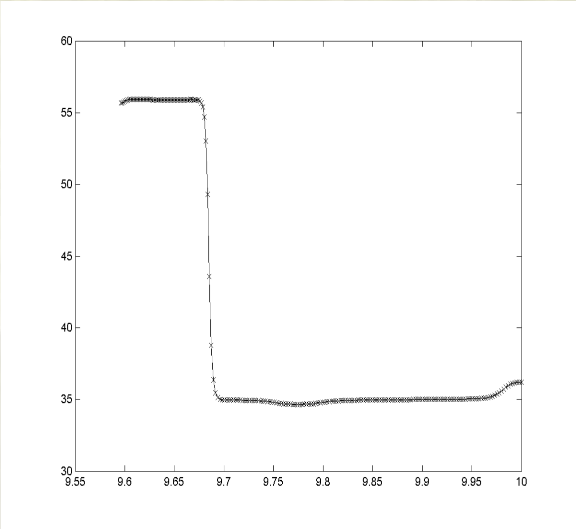
velocity



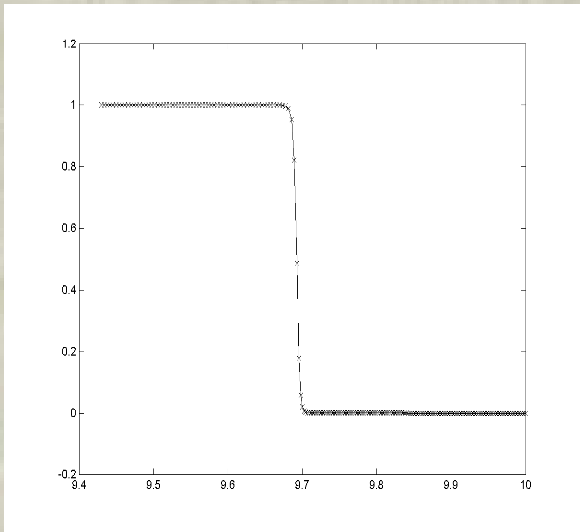


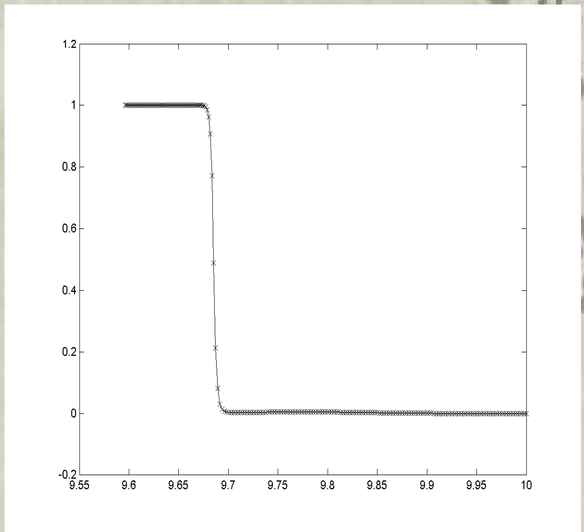
density



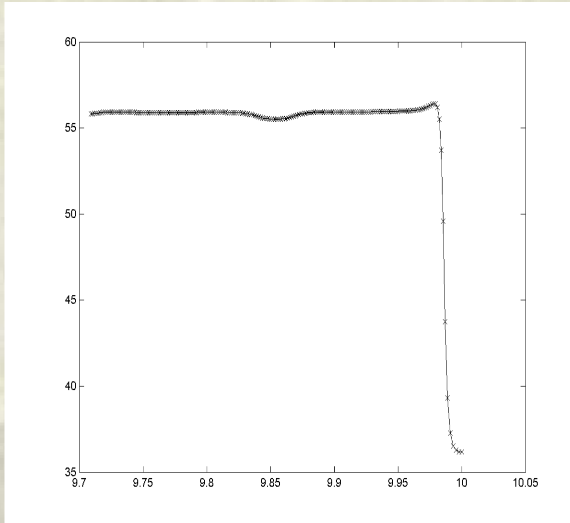


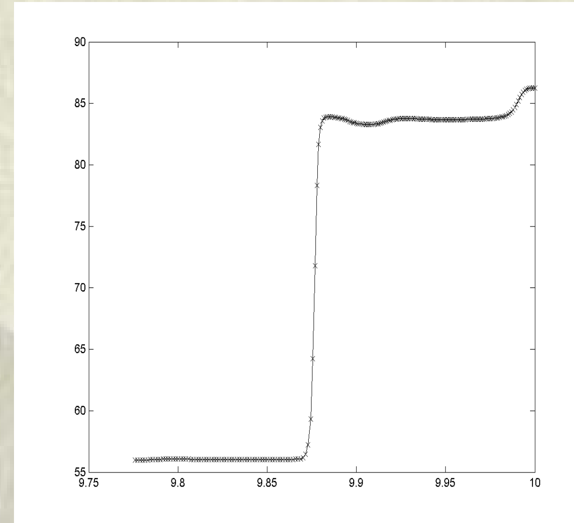
velocity



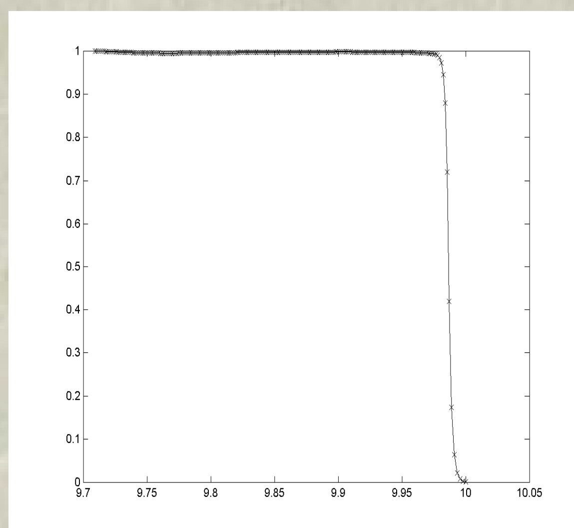


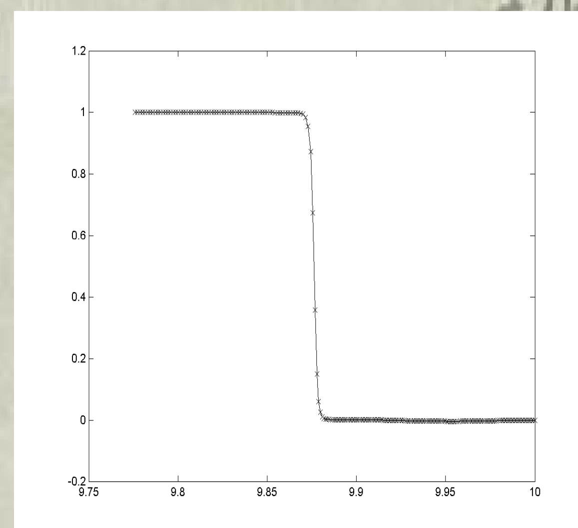
density





velocity





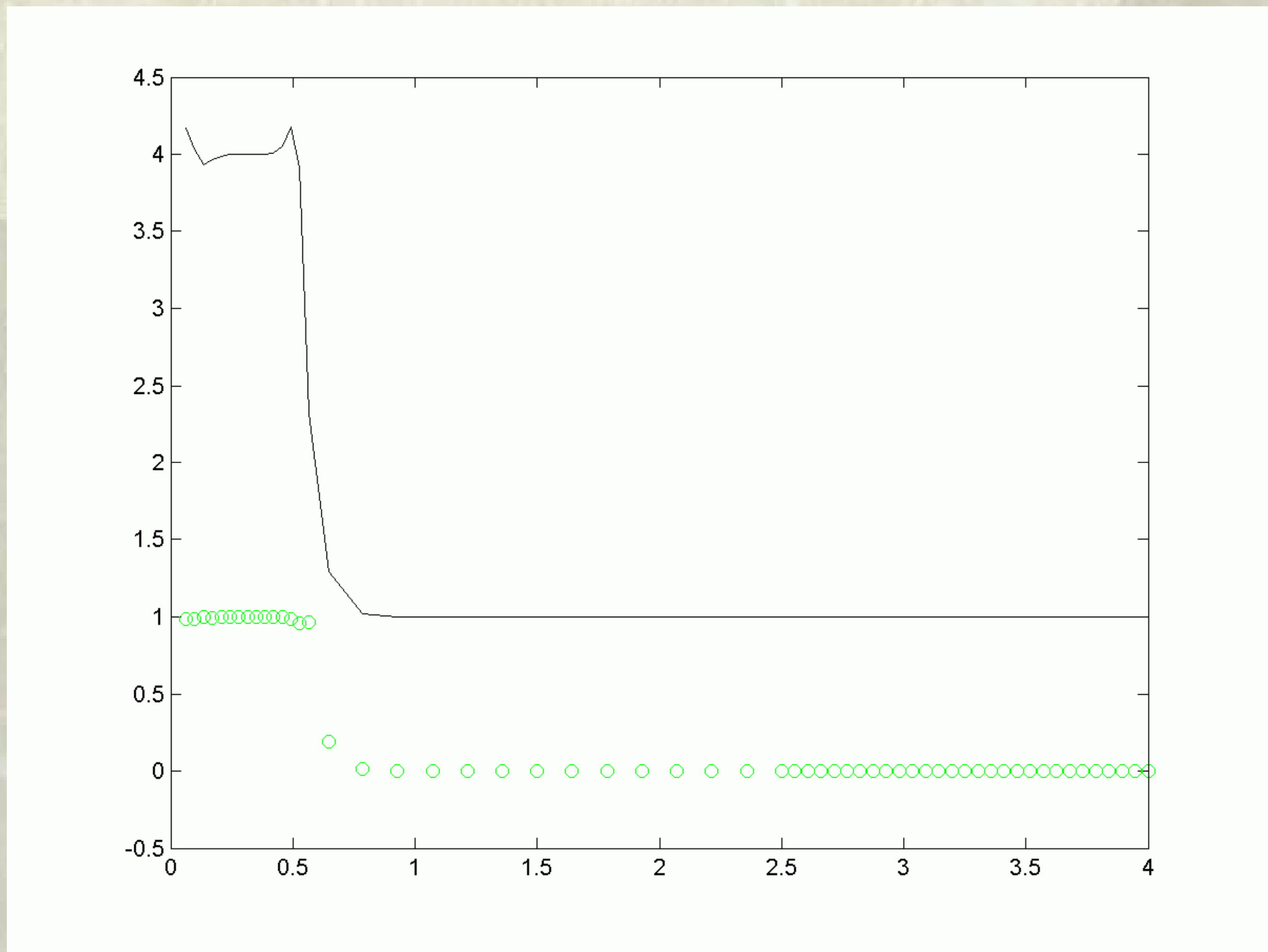
Numerical Results Discussion

- The numerical results show, that the scheme has inherited its good quality from the Euler parent KT scheme
- The smoothness is minimal (even after more, than 10 shock reflections the number of point at the shock is not more then five)
- Some entropy traces are presented (at the boundaries and at the place, where initially the shock has been located)

The internal interface treatment

- KT scheme allows to consider internal matter interfaces
- On the next slide one can see the results of 1D shock problem treatment
- The shock moves through artificial interface, all matter parameters at the right side of interface are equal to the left matter ones.
- The difference is only in the left- and right- cells volumes

Shock wave problem. Black – density, green – velocity.



The internal interface treatment

- The results of calculation, presented at the previous slide, show that entropy traces could appear in the presence of the matter interface
- On the authors point of view a special anti-entropy viscosity should be constructed to damp this non-monotonicity

Difference Scheme for Multidimensional Irregular Grid

- The multidimensional Voronoi grid is used
- To obtain scheme for multi-dimensional case one should reformulate original 1D KT scheme to make generalization more obvious (see the next 2 slides)

Variation for generalization for multidimensional irregular grid.

$$(I \cdot g)_i^{n+1} = (I \cdot g)_i^n - \frac{\tau}{h_i}\left\{F_{i+1/2}^I - F_{i-1/2}^I + V_{i+1/2}^I - V_{i-1/2}^I\right\}$$

$$(E \cdot g)_i^{n+1} = (E \cdot g)_i^n - \frac{\tau}{h_i}\left\{F_{i+1/2}^E - F_{i-1/2}^E + V_{i+1/2}^E - V_{i-1/2}^E\right\}$$

$$x_i^{n+1} = x_i^n + \tau u_i - \frac{\tau V_i^g}{g_i}$$

Flow Calculations

$$F^{I}_{i+1/2} = 0.5 \cdot (p_{i+1} + p_i)$$

$$F^{E}_{i+1/2} = 0.5 \cdot ((pu)_{i+1} + (pu)_i)$$

Viscous Term Calculations

$$V^{I}_{i+1/2} = -\frac{a_{i+1/2}}{2} \cdot \left[I_{i+1} - I_i - \frac{h_{i+1/2}}{2} \cdot (DI_{i+1} + DI_i) \right]$$

$$V^{E}_{i+1/2} = -\frac{a_{i+1/2}}{2} \cdot \left[E_{i+1} - E_i - \frac{h_{i+1/2}}{2} \cdot (DE_{i+1} + DE_i) \right]$$

$$V^{g}_{i+1} = -\frac{a_{i+1/2}}{2} \cdot \left[(\Delta x)_i - (D_2 x)_i \right]$$

MinMod Calculations

$$DI_i = \Theta \cdot MinMod\left(\frac{I_i - I_{i-1}}{h_{i+1/2}}, \frac{I_{i+1} - I_i}{h_{i+3/2}} \right)$$

$$DE_i = \Theta \cdot MinMod\left(\frac{E_i - E_{i-1}}{h_{i+1/2}}, \frac{E_{i+1} - E_i}{h_{i+3/2}} \right)$$

$$(\Delta x)_i = \frac{x_{i+1} - x_i}{h_{i+3/2}} - \frac{x_i - x_{i-1}}{h_{i+1/2}}$$

$$(D_2 x)_i = \frac{\Theta}{2}\left[MinMod\left((\Delta x)_{i-1}, (\Delta x)_i\right) + MinMod\left((\Delta x)_i, (\Delta x)_{i+1}\right) \right]$$

Explanation of Some Details of the Multidimensional Scheme, Presented on the Next Slide.

- Ω_i is a set of numbers of cells, which are neighbor cells to the i-th one
- DI, DE are the results of minmod procedure applied to the gradients (in common case six ones), calculated in the Delauneux triangles, including i-th point
- Dr is a result of minmod operation applied to the corresponding laplacians, calculated in the neighbor points

Lagrangian KT Scheme for Irregular Grid I

$$\frac{\partial \rho_i \cdot g_i x_i^{\nu}}{\partial t} = 0$$

$$\frac{\partial}{\partial t}(\vec{I}_i \cdot g_i x_i^{\nu}) + x_i^{\nu} \sum_{j \in \Omega_i} \frac{1}{2}(p_i + p_j) \cdot \vec{n}_{ij} =$$

$$= \sum_{j \in \Omega_i} \frac{c_{ij}}{2} \left\{ \left(\vec{I}_j - \vec{I}_i \right) \cdot l_{ij} - \frac{1}{2} \left(\left(\vec{n}_{ij} \cdot \vec{D} \right) \cdot \vec{I}_i + \left(\vec{n}_{ij} \cdot \vec{D} \right) \cdot \vec{I}_j \right) \right\} \cdot x_{ij}^{\nu}$$

$$\frac{\partial}{\partial t}\left(E_i \cdot g_i x_i^{\nu} \right) + \sum_{j \in \Omega_i} \frac{1}{2} \left((p\vec{u})_i + (p\vec{u}_j) \right) \cdot \vec{n}_{ij} \cdot x_{ij}^{\nu} =$$

$$= \sum_{j \in \Omega_i} \frac{c_{ij}}{2} \left\{ \left(E_j - E_i \right) \cdot l_{ij} - \frac{1}{2} \left(\left(\vec{n}_{ij} \cdot \vec{D} \right) \cdot E_i + \left(\vec{n}_{ij} \cdot \vec{D} \right) \cdot E_j \right) \right\} \cdot x_{ij}^{\nu}$$

$$\frac{\partial}{\partial t} \vec{r}_i = \vec{u}_i + \frac{1}{g_i} \left(\Delta \vec{r}_i - D \vec{r}_i \right)$$

$$\Delta \vec{r}_i = \sum_{j \in \Omega_i} \frac{c_{ij}}{2} (\vec{r}_j - \vec{r}_i) \cdot l_{ij}$$

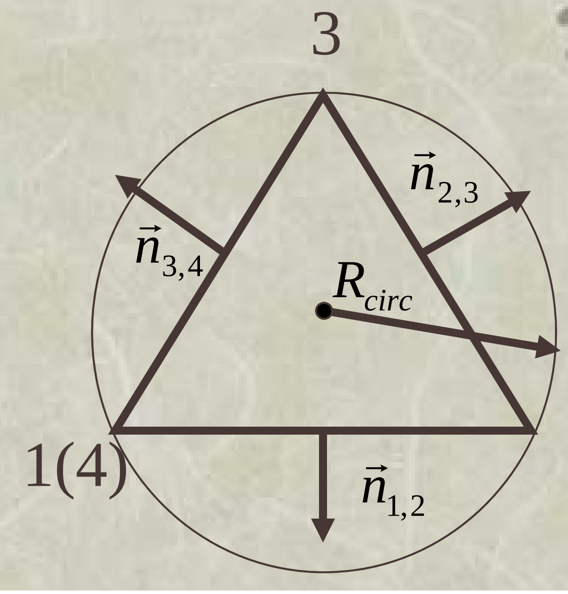
Lagrangian KT Scheme for Irregular Grid II

Gradient of functions:

$$\vec{\nabla}\phi_{123} = \frac{R_{circ}}{S}\cdot\sum_{i=1}^{3}\left(\phi_i + \phi_{i+1}\right)\frac{\vec{n}_{i,i+1}}{\left|\vec{n}_{i,i+1}\right|} = \frac{R_{circ}}{S}\cdot\sum_{i=1}^{3}\left(\phi_i + \phi_{i+1}\right)\cdot\vec{n}^{e}_{i,i+1}$$

$$\vec{\nabla}\vec{\phi}_{123} = \begin{pmatrix}\vec{\nabla}\phi^{x}_{123}\\ \vec{\nabla}\phi^{y}_{123}\end{pmatrix} = \frac{R_{circ}}{S}\cdot\begin{vmatrix}\sum_{i=1}^{3}\left(\phi^{x}_i + \phi^{x}_{i+1}\right)\\ \sum_{i=1}^{3}\left(\phi^{y}_i + \phi^{y}_{i+1}\right)\end{vmatrix}\cdot\vec{n}^{e}_{i,i+1}$$

R_{circ} – radius of circumscribed circle



Lagrangian KT Scheme for Irregular Grid III

MinMod calculations:

$$\vec{D}\phi_i = \min_{j\in\Omega_i,\; k\in\Omega_j\cap\Omega_i} \bmod\left\{\vec{\nabla}\phi_{ijk}\right\} = \begin{pmatrix} D\phi^x_{ijk} \\ D\phi^y_{ijk} \end{pmatrix}$$

$$D\vec{r}_i = \min_{j\in\Omega_i} \bmod\left\{\Delta\vec{r}_i, \Delta\vec{r}_j\right\} = \begin{pmatrix} Dx_i \\ Dy_i \end{pmatrix}$$

$$\min_{i\in\Omega} \bmod\{u_i\} = \begin{cases} 0, & if\ \ \exists i, j \in \Omega : sign(u_i) \neq sign(u_j) \\ \min_{i\in\Omega}\{|u_i|\}\cdot sign(u_i) \end{cases}$$

$$\min_{i\in\Omega} \bmod\{\vec{u}_i\} = \left(\min_{i\in\Omega} \bmod\left\{u_i^x\right\}, \min_{i\in\Omega} \bmod\left\{u_i^y\right\}\right)$$

Lagrangian KT Scheme for Irregular Grid IV

Coordinates:

$$x_i^{n+1} = x_i + \tau^n \cdot \left(u_i^x + \frac{1}{g_i} \left(\Delta x_i - Dx_i \right) \right)$$

$$y_i^{n+1} = y_i + \tau^n \cdot \left(u_i^y + \frac{1}{g_i} \left(\Delta y_i - Dy_i \right) \right)$$

$$\Delta x_i = \sum_{j \in \Omega_i} \frac{c_{ij}}{2} \left(x_j - x_i \right) \cdot l_{ij}$$

$$\Delta y_i = \sum_{j \in \Omega_i} \frac{c_{ij}}{2} \left(y_j - y_i \right) \cdot l_{ij}$$

Lagrangian KT Scheme for Irregular Grid V

Velocities:

$$u_i^{x^{n+1}} = u_i^x - \frac{\tau^n}{2 \cdot \rho_i \cdot v_i} \sum_{j \in \Omega_i} \begin{pmatrix} x_i^\nu \cdot \left(p_i + p_j\right) \cdot n_{ij}^x + \\ \begin{pmatrix} \frac{1}{2}\left(\left(DI_i^{xx} + DI_j^{xx}\right) \cdot n_{ij}^x + \left(DI_i^{xy} + DI_j^{xy}\right) \cdot n_{ij}^y\right) - \\ c_{ij} \cdot \left(\rho_j \cdot u_j^x - \rho_i \cdot u_i^x\right) \cdot l_{ij} \end{pmatrix} \cdot x_{ij}^\nu \end{pmatrix}$$

$$u_i^{y^{n+1}} = u_i^y - \frac{\tau^n}{2 \cdot \rho_i \cdot v_i} \sum_{j \in \Omega_i} \begin{pmatrix} x_i^\nu \cdot \left(p_i + p_j\right) \cdot n_{ij}^y + \\ \begin{pmatrix} \frac{1}{2}\left(\left(DI_i^{yx} + DI_j^{yx}\right) \cdot n_{ij}^x + \left(DI_i^{yy} + DI_j^{yy}\right) \cdot n_{ij}^y\right) - \\ c_{ij} \cdot \left(\rho_j \cdot u_j^y - \rho_i \cdot u_i^y\right) \cdot l_{ij} \end{pmatrix} \cdot x_{ij}^\nu \end{pmatrix}$$

Lagrangian KT Scheme for Irregular Grid VI

Energy:

$$E_i^{n+1} = E_i - \frac{\tau^n}{2\cdot v_i}\sum_{j\in\Omega_i}\left(\begin{pmatrix} \left(\left(p_i \cdot u_i^x + p_j \cdot u_j^x\right)\cdot n_{ij}^x + \left(p_i \cdot u_i^y + p_j \cdot u_j^y\right)\cdot n_{ij}^y\right) + \\ \frac{1}{2}\left(\left(DE_i^x + DE_j^x\right)\cdot n_{ij}^x + \left(DE_i^y + DE_j^y\right)\cdot n_{ij}^y\right) - \\ c_{ij}\cdot\left(E_j - E_i\right)\cdot l_{ij} \end{pmatrix}\cdot x_{ij}^{\nu}\right)$$

where

$$\vec{n}_{ij} = \begin{pmatrix} n_{ij}^x \\ n_{ij}^y \end{pmatrix};$$

$$\vec{u}_i = \left(u_i^x, u_i^x\right); \; v_i = g_i \cdot x_i^{\nu}$$

$$D\vec{r}_i = \left(Dx_i, Dy_i\right); \quad \vec{D}E_i = \left(DE_i^x, DE_i^x\right);$$

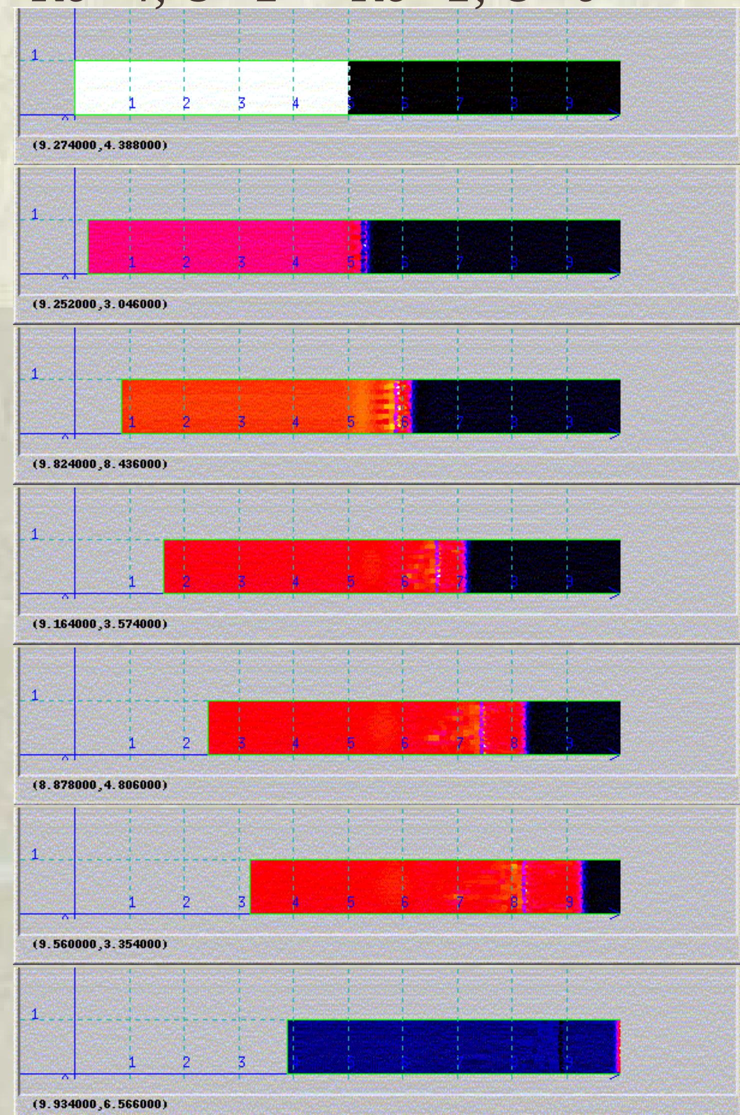
$$\vec{D}\vec{I}_i = \begin{pmatrix} DI_i^{xx}, DI_i^{xy} \\ DI_i^{yx}, DI_i^{yy} \end{pmatrix}$$

A test problem for the multidimensional irregular grid in 1D case

- In the next slide results of a test problem numerical modeling are presented.
- The initial conditions are: the first region – density 4, specific internal energy – 0.5, velocity – 1; the second region – density – 1, specific internal energy – 1e-6, velocity – 0.
- Boundary conditions: the left boundary – piston with velocity 1, the right one – rigid wall.

1D Shock Wave for 2D Irregular Grid. Density.

Ro=4, U=1 *Ro=1, U=0*



t=0.000

t=0.241

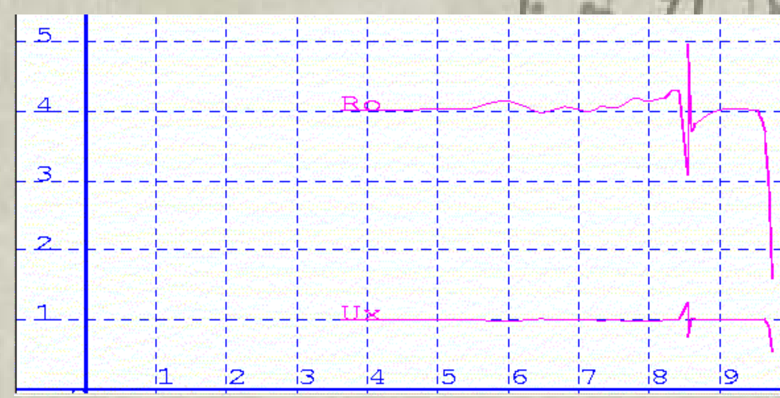
t=0.855

t=1.609

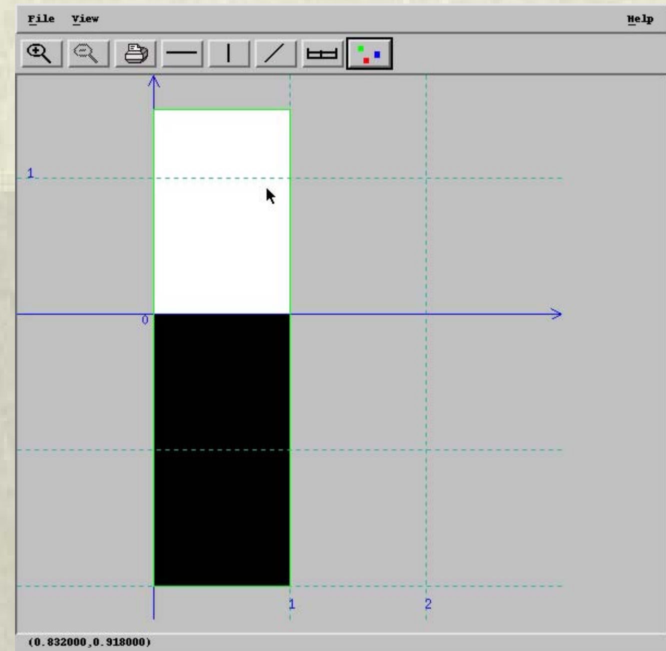
t=2.431

t=3.209

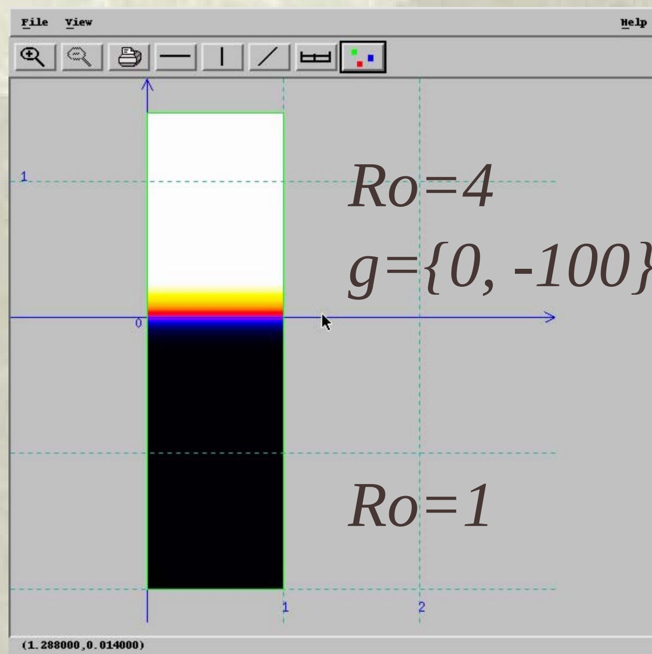
t=3.887



RT Instability I.

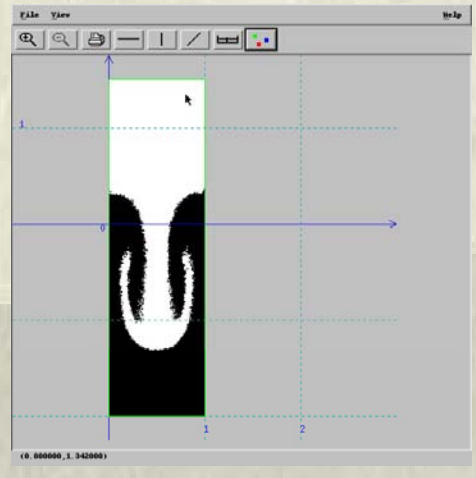
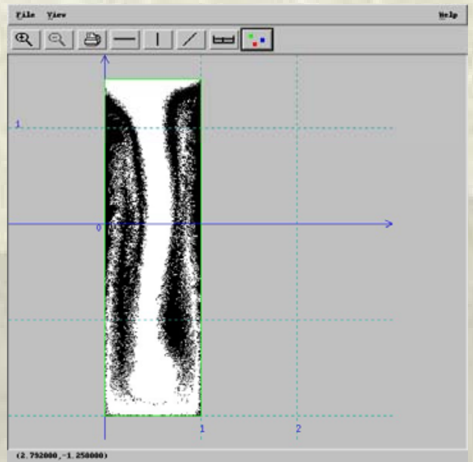
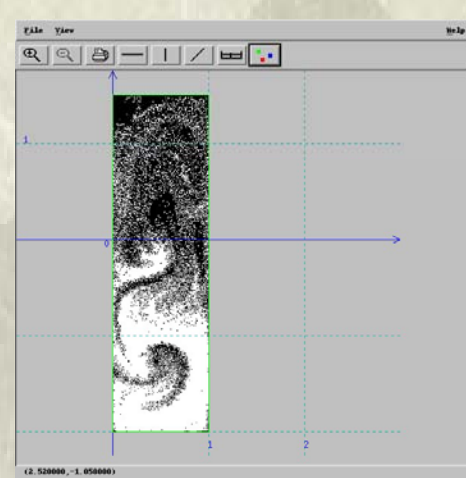


Material

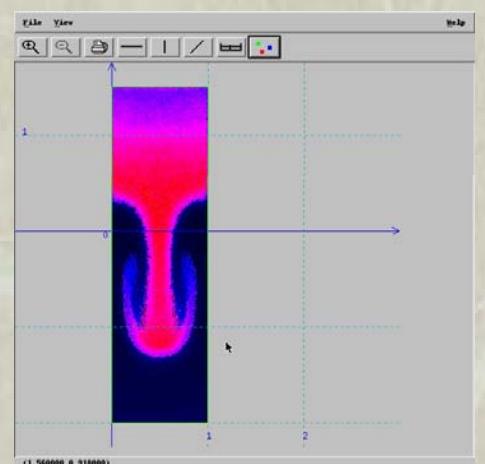
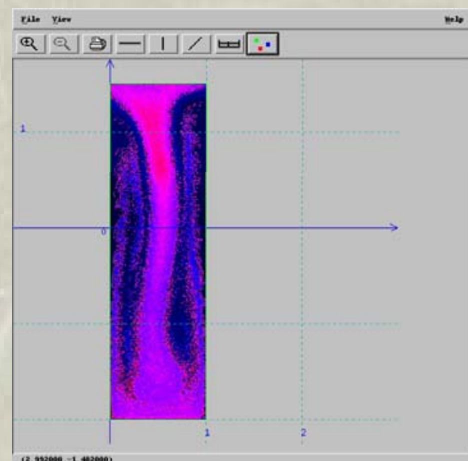
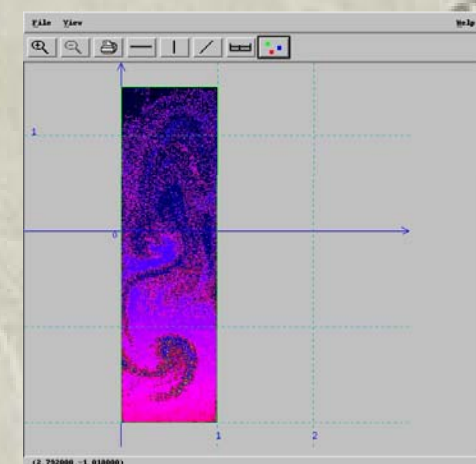


Density

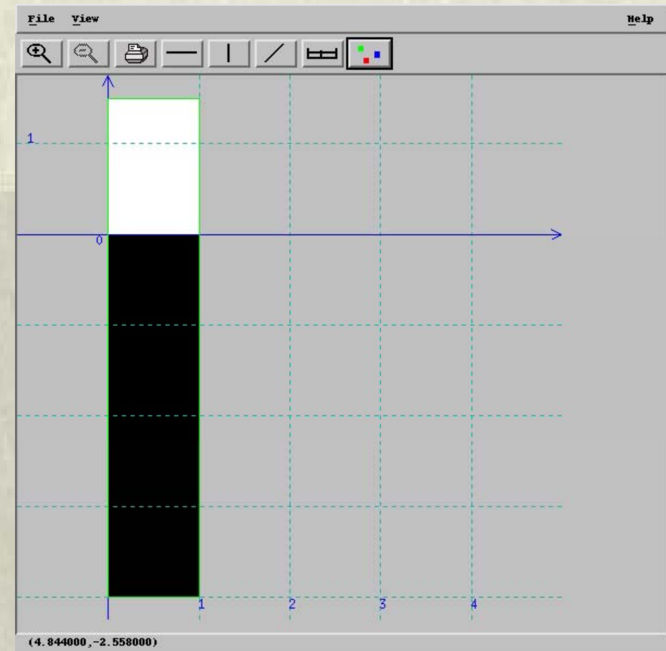
RT Instability II.

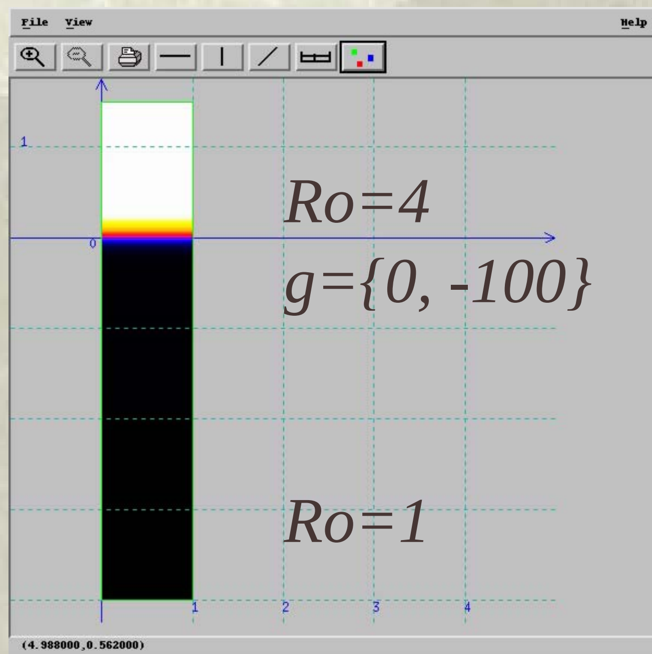
t=0.440 *t=0.920* *t=2.200*

RT Instability III.

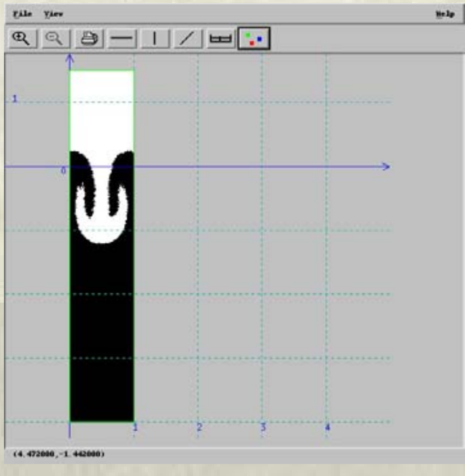


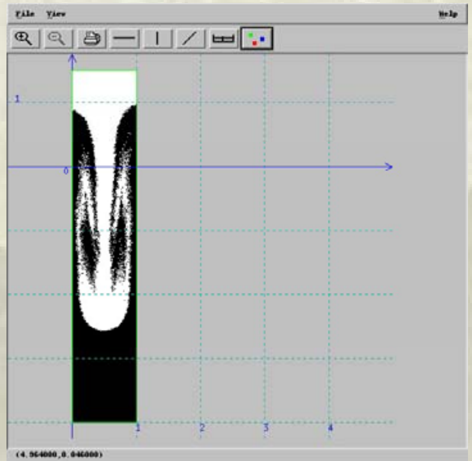
Material

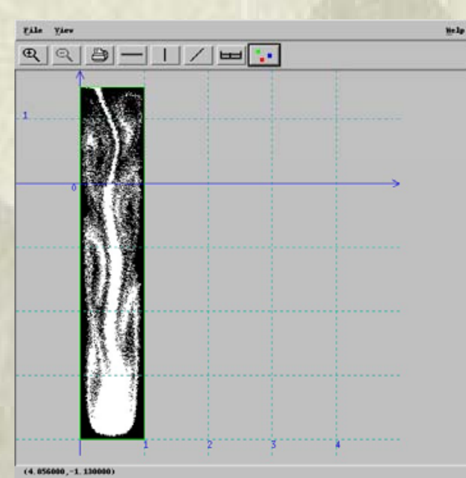


Density

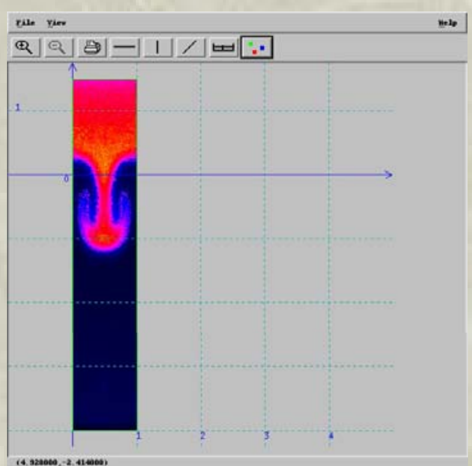
RT Instability IV.

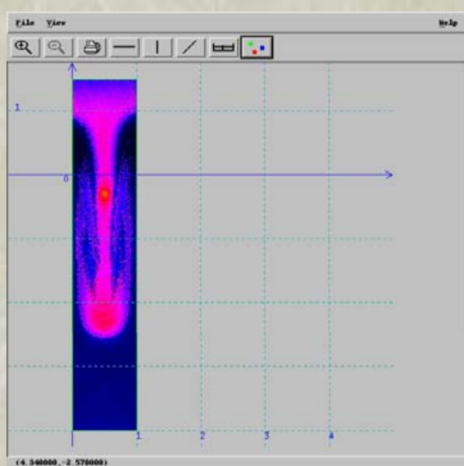


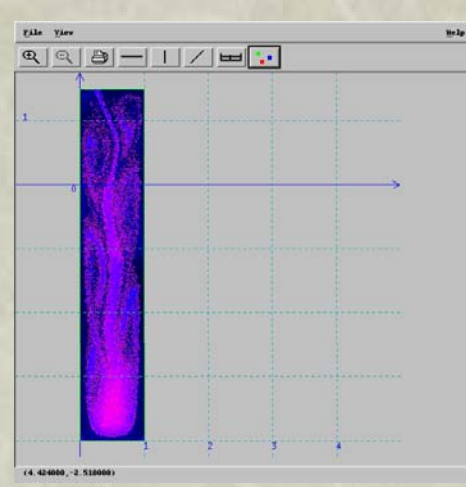




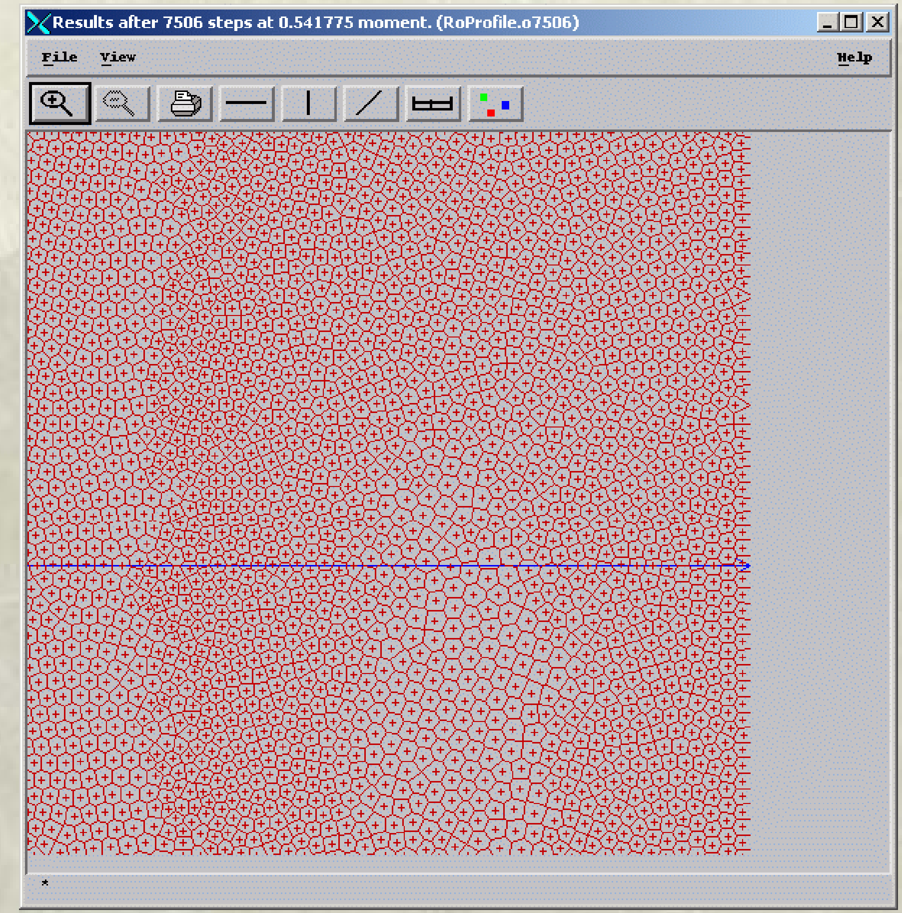
t=0.436 *t=0.892* *t=1.645*







RT Instability V.



Irregular Grid